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# ON MULTIDIMENSIONAL NON-GAUSSIAN RANDOM FIELDS GENERATION FOR FINITE ELEMENT STRUCTURAL ANALYSIS.

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***Abstract.*** *Random fields generation is a very important subject related with stochastic finite element structural analysis and structural reliability. In these cases, despite of the inherent randomness of the different variables, the designer is concerned with the spatial randomness of material properties, geometry, applied external loads and boundary conditions in order to improve the representation of the system characteristics. In this way, the multidimensional non-Gaussian stochastic field generation becomes, in many cases, an appropriated tool to obtain reliable results.*
*A brief review of three models for random fields generation is presented in this work. In the three models the inverse mapping technique is used to obtain a non-Gaussian field. The Cholesky Decomposition method has been used intensively for any field correlation. However, through the modal decomposition method, the decreasing characteristics of the covariance matrix eigenvalues is used, reducing significantly the computational effort and cost to generate random fields. Finally, the spectral representation method, employing cosine series, is very useful to obtain accurate generations of random fields due to orthogonality and periodicity of the adopted trigonometric functions.*
*Numerical examples for random field generation are presented for a quantitative and qualitative evaluation of these methods, having in sight future applications in the reliability analysis of structural systems.*

***keywords:*** *Random fields generation, Stochastic finite element analysis.*

## 1. INTRODUCTION

Random fields generation is a very important subject related with stochastic finite element structural analysis and structural reliability. In these cases, despite of the inherent randomness of the different variables, the designer is concerned with the spatial randomness of material properties, geometry, applied external loads and boundary conditions in order to improve the representation of the system characteristics. In the reliability analysis of three-dimensional structures, the generation of such fields is of great importance, mainly in the determination of geometry and loads, since one wants to work with models that best represent the actual structures. Due to mathematical difficulties and the lack of observed data, most of the generation techniques as well as analytic approaches for stochastic fields generation are limited to the treatment of Gaussian fields. In some situations, the hypothesis of a stochastic Gaussian field is not adequate due to the fact that the observed actual fields present non-Gaussian characteristics. As exemplified in Yamazaki *et al.*(1988), the answer of non-linear systems is known as non-Gaussians, even for Gaussians inputs. Also, for example, the space variability of the Young modulus and other mechanical properties should not be considered as Gaussians, because theoretically these properties do not assume negative values. It is more appropriate to assume them as stochastic fields with a lognormal distribution.

### 1.1 Stochastic field generation

The digital generation technique of stochastic fields has been established in the last two decades (Yamazaki *et al.*, 1988). In general, the simulation and generation of samples of stochastic fields can be obtained by means of: (a) spectral representation; (b) ARMA modeling (Auto-Regressive Moving Average); and (c) covariance matrix decomposition procedures (Yamazaki *et al.*, 1990).

Yamazaki *et al.* (1990) shows that with the spectral representation, stochastic fields can be simulated and the respective samples generated, and that, due to periodicity and orthogonality of trigonometric functions used in the expansion, the resulting spatial statistics are highly accurate. The digital generation of the samples can also be carried out efficiently with the aid of the FFT (Fast Fourier Transform) algorithm. The ARMA model for stochastic fields representation has been received considerable attention recently. Basically, the advantage in use of the ARMA model lies in the small amount of requested memory and CPU time for the field generation due to the recursive form involved in the formulation. As was observed by Yamazaki *et al.*, (1990), this method usually requests a very large number of samples, so that the same level of statistical precision is reached, for example, with the method of spectral representation which requests a smaller number of samples.

The covariance matrix decomposition methods, such as Cholesky decomposition and modal decomposition, allow the generation of stochastic fields in discrete points, since their correlation functions, or optionally their spectral density functions, are given *a priori*. The great advantage of these methods lies in the fact that the generation of multivariate, multidimensional, nonhomogencous and non-Gaussians stochastic fields is as easily as for unidimensional ones. However, the covariance decomposition matrix method could request a great amount of samples to achieve high accurate and stable statistics. In the following paragraphs, a brief description in use of covariance matrix decomposition methods is given for the generation of non-Gaussian isotropic stochastic fields.

## 2. NON-GAUSSIAN FIELDS GENERATION

The stochastic fields generation of non-Gaussian distributions follows the inverse mapping technique used in random single variable generation with prescribed probability density function. As well as in the case of random single variables, it is necessary to generate Gaussian variables with zero mean and unit standard deviation. Then, it is correlated to this space. The inverse mapping technique is then used to impose the correct probability density function.

One must starts by generating uniformly distributed random numbers ($\underset{\sim}{u}$) between $0$ and $1$ in the same amount of the discrete field. Then, the inverse cumulative standard Gaussian distribution function is used to obtain a standard Gaussian field (zero mean and unit standard deviation, $\underset{\sim}{Z'}$) as described by equation (1):

$$\underset{\sim}{Z'} = \Phi^{-1}(\underset{\sim}{u}) \tag{1}$$

where $\Phi^{-1}$ is the inverse cumulative standard Gaussian distribution function.

In statistical analysis of actual stochastic fields, the obtained statistics are regarding to the actual space variable. Thus, the available data concerning to a given variable are its distribution function type and autocorrelation matrix. If, for example, a variable has a lognormal distribution function type, then a description in terms of its mean, standard deviation and autocorrelation matrix ($R_{X\,ij}$, in the actual space) is enough.

The Nataf model may be used for the covariance matrix transformation to the non-correlated standard Gaussian space as described in Liu *et al.*, (1986). Basically, given the correlation coefficient matrix in the actual space ($\rho_{ij}$), the correlation coefficient matrix in the standard Gaussian space ($\rho'_{ij}$) can be found through the numerical solution of the integral given below:

$$\rho_{ij} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{x_i-\mu_i}{\sigma_i}\right)\left(\frac{x_j-\mu_j}{\sigma_j}\right)\varphi_2\left(z_i, z_j \rho'_{ij}\right) dz_i\ dz_j \tag{2}$$

where $\mu_i$ and $\sigma_i$ they are the mean and standard deviation of the $i$ th. variable and $\varphi_2\left(z_i, z_j \rho'_{ij}\right)$ is the bidimensional normal probability density function of zero means and unit standard deviations. As the correlation coefficients are inside the integral, numerical integration or even empirical formulae may be used to solve this kind of problem (Liu *et al.*, 1986).

As we do not know $\rho'_{ij}$, it is necessary to transform the given autocorrelation matrix ($R_{X\,ij}$) in the correlation coefficient matrix ($\rho'_{ij}$) in order to use the Nataf model. Then, following the definition of the correlation coefficient matrix, and reminding that homogeneity and a zero mean value for the stochastic field were assumed, one can write:

$$C_{X\,ij} = R_{X\,ij} \text{ and } \rho_{ij} = \frac{R_{X\,ij}}{\sigma_{X_i}\sigma_{X_i}} \tag{3}$$

where $C_{X\,ij}$ is the covariance matrix, $R_{X\,ij}$ is the autocorrelation matrix and $\sigma_{X_i}$ is the standard deviation of $i$ th. variable.

Once the correlation coefficient matrix ($\underset{\sim}{\rho}$) in the actual space was obtained and the correlation coefficient matrix in standard Gaussian space ($\underset{\sim}{\rho'}$) was calculated, two methods for matrix decomposition could be used as indicated below:

- Cholesky Decomposition: (Rippley , 1987)

$$\underset{\sim}{\rho'} = \underset{\sim}{L}\,\underset{\sim}{L}^T \tag{4}$$

where $\underset{\sim}{L}$ is a lower triangle matrix.

- Modal Decomposition:

$$\underset{\sim}{\rho'} = \underset{\sim}{\Theta}\,\underset{\sim}{\Lambda}\,\underset{\sim}{\Theta}^T \tag{5}$$

where $\underset{\sim}{\Lambda}$ is the eigenvalues diagonal matrix in descending order and $\underset{\sim}{\Theta}$ is the eigenvectors matrix associated to the eigenvalue-eigenvector problem:

$$\underset{\sim}{\rho'}\,\underset{\sim}{\Theta} = \underset{\sim}{\Theta}\,\underset{\sim}{\Lambda} \tag{6}$$

The following three procedures are applicable to generate prescribed standard Gaussian correlated variables ( $\underset{\sim}{Z}$ ):

- Multiplying the lower triangle matrix from Cholesky decomposition by the standardized Gaussian numbers (Cholesky decomposition method):

$$\underset{\sim}{Z} = \underset{\sim}{L}\,\underset{\sim}{Z'} \tag{7}$$

- Multiplying the eigenvector matrix (or part of it) by the square root of the eigenvalues and by the standard Gaussian numbers, as shown in equation (8) (Modal decomposition method):

$$\underset{\sim}{Z} = \underset{\sim}{\Theta}\,\underset{\sim}{\Lambda}^{1/2}\,\underset{\sim}{Z'} \tag{8}$$

- Direct generation of standard Gaussian numbers by the so called spectral representation using the eigenvector and eigenvalue matrices from the previous procedure and cosine series (Shinozuka *et al.*, 1985), as indicated in equation (9) (Spectral representation by cosine series method):

$$\begin{gathered}\underset{\sim}{Z} = Z_s = \sqrt{2}\sum_{i=1}^{M}\sum_{m=1}^{N_f}\Theta_{si}\sqrt{\frac{\lambda_i}{N_f}}\cos\left(\frac{\pi}{2}\frac{T_{nN_f}}{T_k}+\psi_k\right) \\ (s=1,2,...,n) \quad (k=1,2,...,nN_f) \quad k=(m-1)M+i\end{gathered} \tag{9}$$

where $\lambda_i$ is $i$ th. eigenvalue of the correlation coefficient matrix (in descending order), $\Theta_{si}$ is the $s$ th. component of the vector $\Theta_i$, $\psi_k$ is the $k$ th. uniformly distributed phase angle between $0$ and $2\pi$ ($\underset{\sim}{\psi} = 2\pi\,\underset{\sim}{Z'}$), $M$ is the number of retained modes of the eigenvector matrix,

$N_f$ is the number of cosine functions to be added, $n$ is the size of the discretized field and $T_k = 1/k$.

Once the correlated standard Gaussian field (with zero mean and unit standard deviation) was found, these values are mapped into the actual field space through the cumulative density function of the random filed, originating:

$$\underset{\sim}{X} = F_X^{-1}(\underset{\sim}{Z}) \tag{10}$$

where $F_X^{-1}$ and $\underset{\sim}{X}$ are the desired inverse cumulative density function and the field respectively.

## 3. NUMERICAL EXAMPLE

In this example, a bidimensional lognormal stochastic field in a steel square plate with dimensions 10,0 x 10,0 m is generated. The square plate is discretized in 100 square elements of equal area (1,0 x 1,0 m). A sketch of the steel square plate is depicted in figure 1. The properties of the element are considered constant along the element area and represented by this value at its center, assuming an autocorrelation function with isotropic exponential form as shown in equation (11):

$$R_{XX}\left(\underset{\sim}{\xi}\right) = \sigma^2 \exp\left[-\left(\left|\underset{\sim}{\xi}\right| / d\right)^2\right] \tag{11}$$

where $\underset{\sim}{\xi}$ is the vector of separation, which contains the center to center distances between elements, $d$ is the parameter representing the correlation scale (the larger $d$, the more slowly the correlation disappears as a function of the separation distance $\underset{\sim}{\xi}$), and $\sigma$ is the stochastic field standard deviation. In the present study the mesh discretization is assumed small enough to represent the random variability. For a more accurate discussion on this problem see, for example, Liu *et al.* (1993).

In this example, the stochastic field represents the spatial variability of the yield stress ($f_y$) on the steel square plate. It is taken $\mu = 500\ MPa$, $\sigma = 30\ MPa$ with lognormal probability density function and $d = 2$. In order to validate the results, the covariance matrix has been calculated from a series of 500 samples generated according to the above-mentioned procedures, assuming isotropy and ergodicity of the generated fields. In figures 2,3,4,5 the original autocorrealtion function, and results of the three methods are compared.

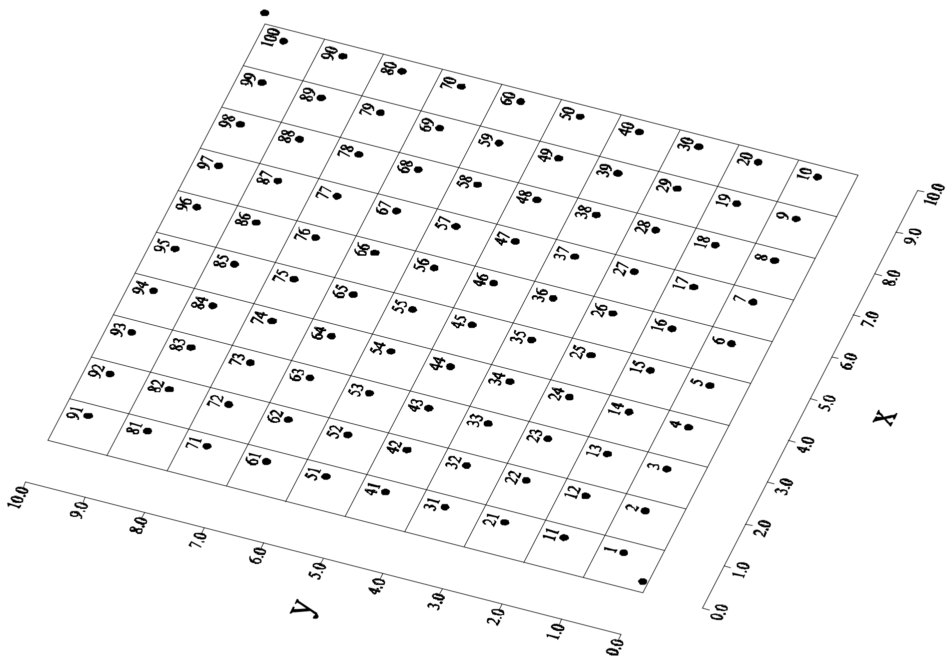

Figure 1 - Square plate discretization  in 100 equal finite elements.

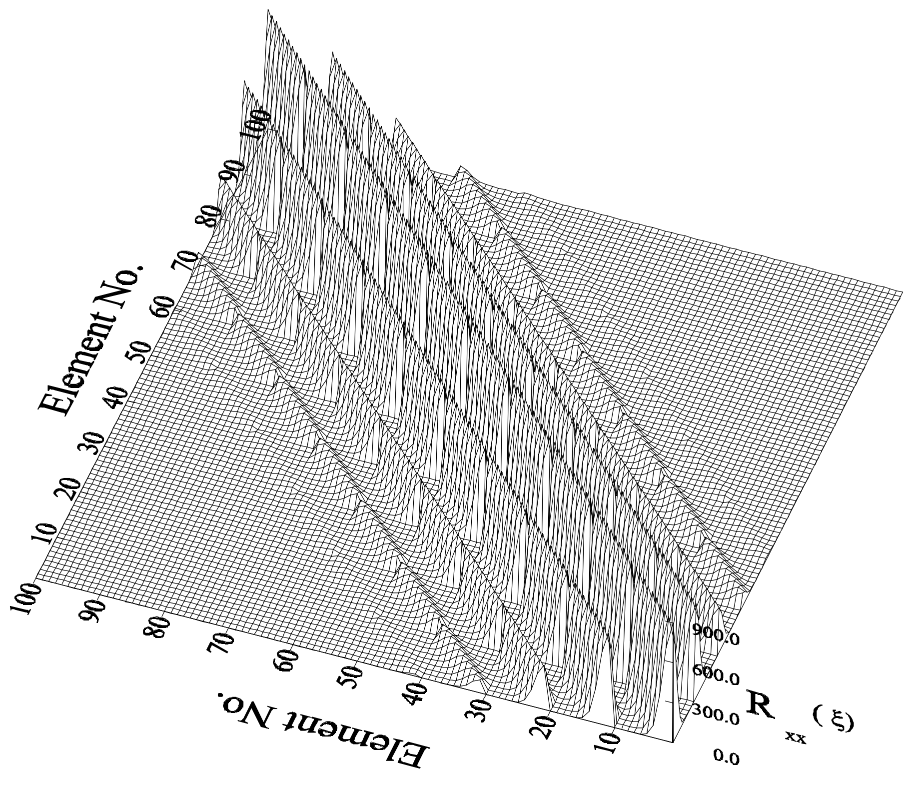


Figure 2 - Original autocorrelation matrix.

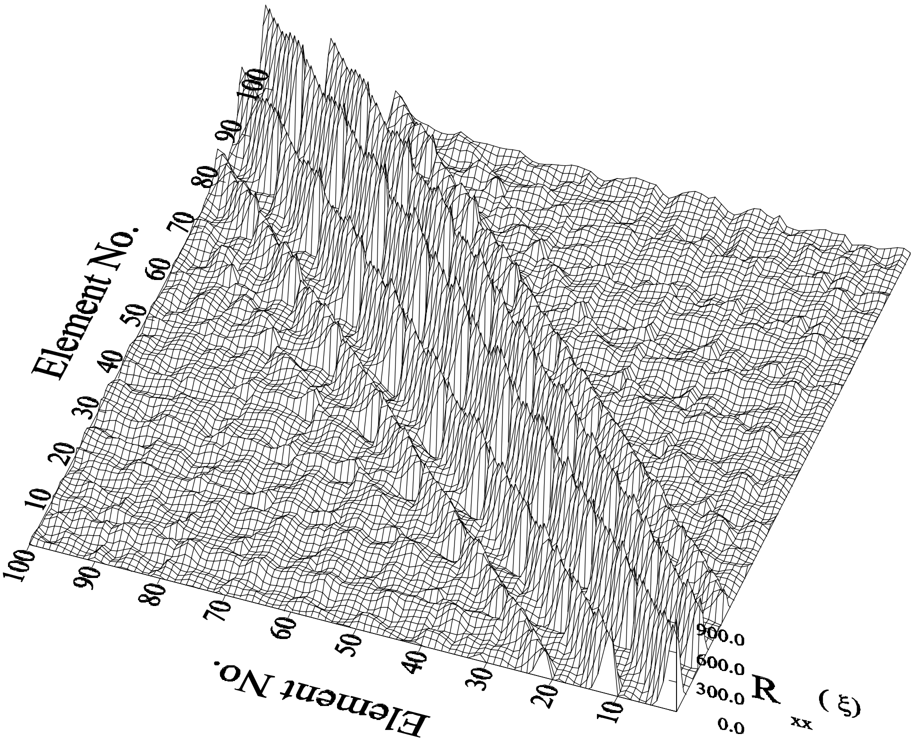


Figure 3 - Autocorrelation function for Cholesky decomposition.

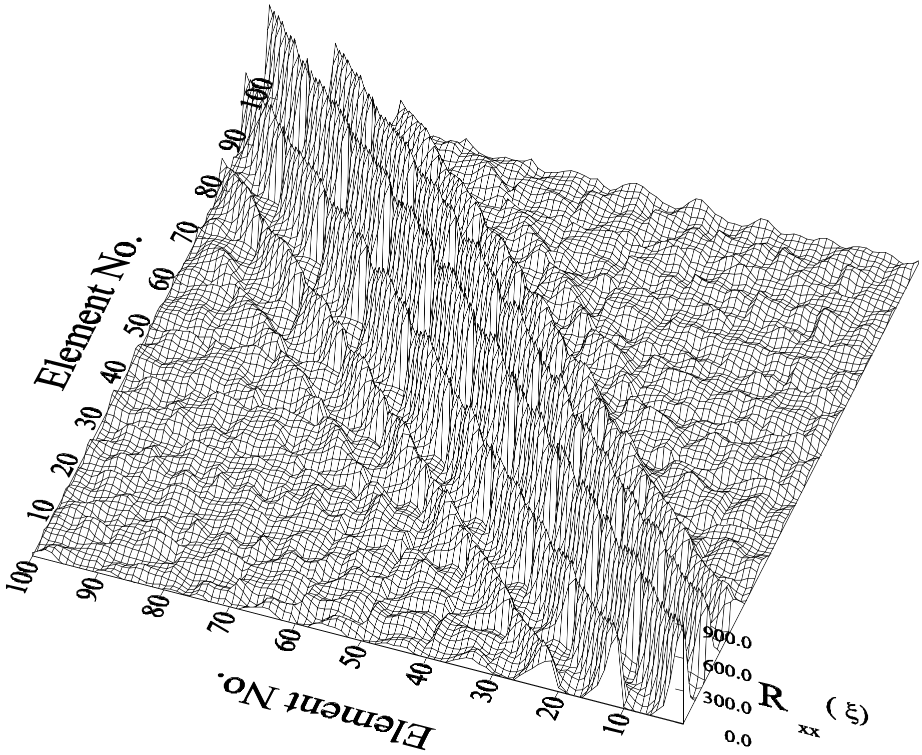


Figure 4 - Autocorrelation function for modal decomposition.

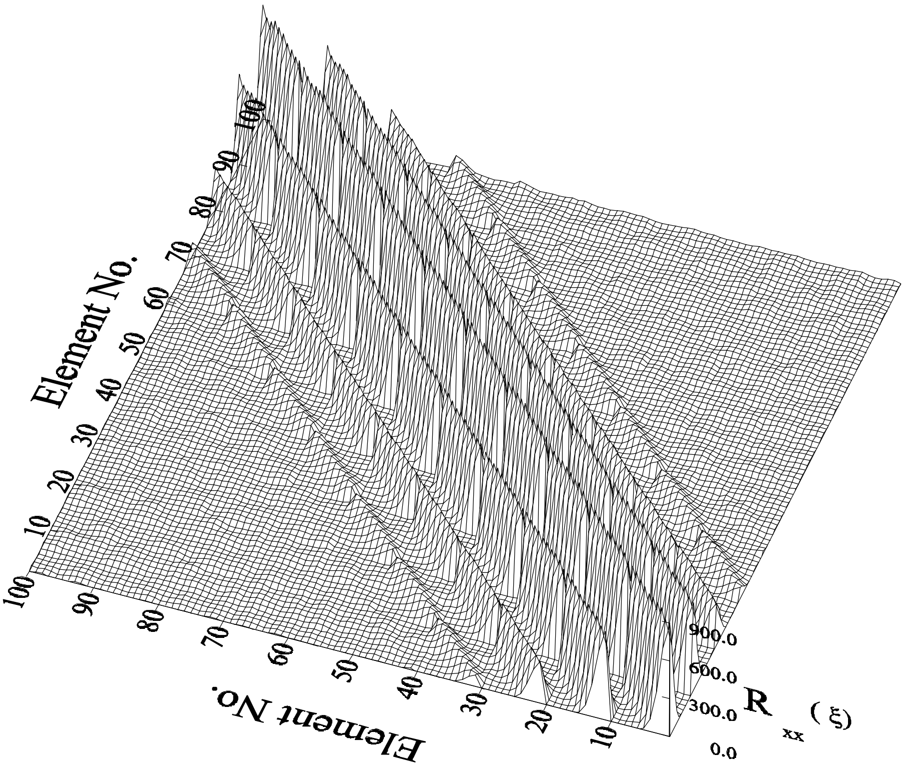


Figure 5 - Autocorrelation function for spectral representation.

In the example presented here, for the spectral representation method it was taken $M = 50$, (the first $50$ modes), and $N_f = 1$ (only one cosine is added in the series). Comparisons could be established among the different formulations. The Cholesky decomposition method seems to be very attractive when the stochastic field is very little correlated, or the out of diagonal elements of the correlation coefficient matrix are very small when compared with diagonal elements. On other hand, the modal decomposition method seems to be more advantageous when Cholesky decomposition do not work properly (Cholesky factorization algorithm fails), i.e., when the field is highly correlated, or the out of diagonal elements of the correlation coefficient matrix are of the same magnitude of the diagonal terms. Since the spectral representation method depends on the eigenvectors and eingenvalues of correlation coefficient matrices, this method has the same advantages and shortcoming as modal decomposition. However it must be emphasize that it is not necessary evaluate all eigenvalues/eigenvectors. Since majority of actual fields are moderately correlated, it is necessary only some modes for an accurate generation.

In figure 6 the corresponding generated fields are shown.

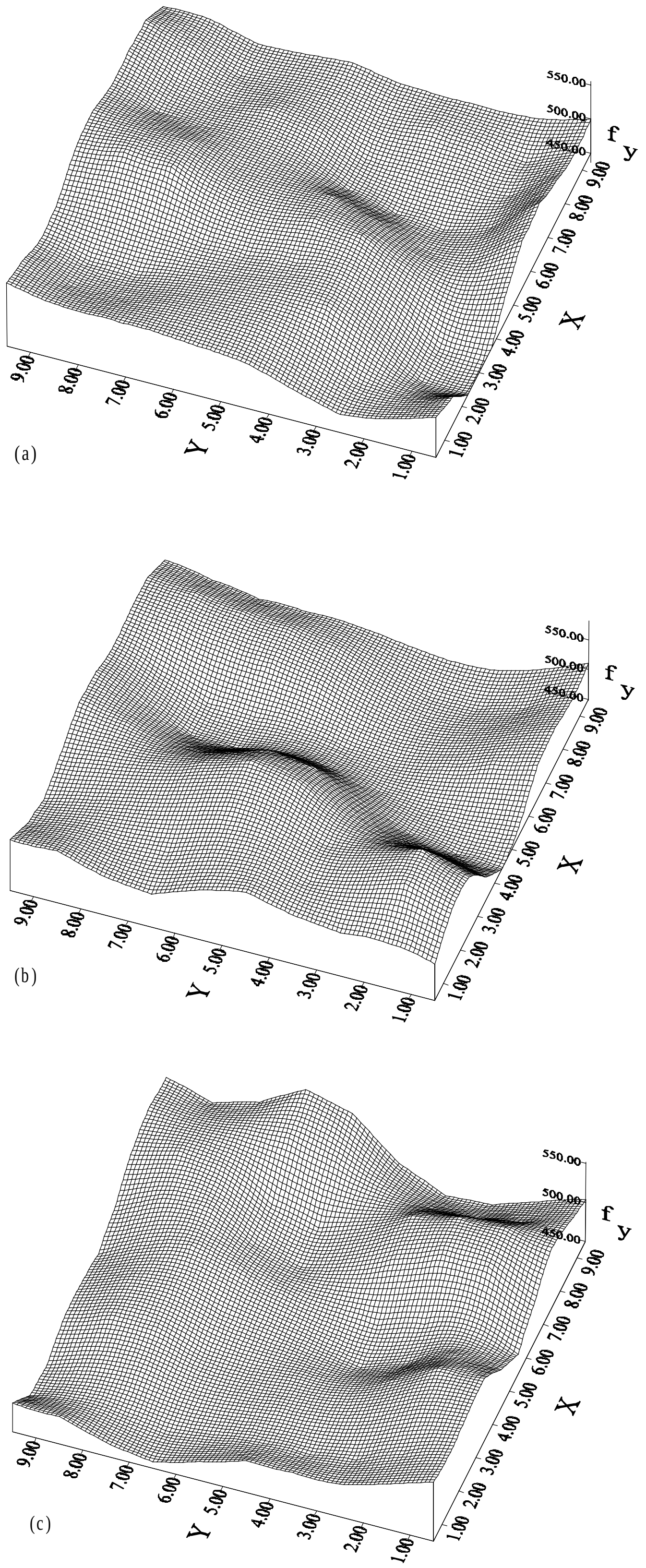


Figure 6 - Generated fields: (a)Cholesky decomposition; (b)Modal decomposition and (c)Spectral representation by cosine series.

The relative CPU time required for each one of the different methods, for three samples sizes, to generate an stochastic fields is shown in Table 1 below:

Table 1 - Relative CPU time for different methods and sample sizes.

| Number of Samples | 500 | 1000 | 2000 |
|---|---|---|---|
| Method | | | |
| Cholesky Decomp. | 1.00 | 1.62 | 3.14 |
| Modal Decomp. | 1.77 | 2.70 | 4.60 |
| Spectral Repres. | 1.18 | 1.18 | 1.22 |

## 4. CONCLUSION

Analyzing figures 2,3,4,5, it is possible notice the more accurate evaluation of the autocorrelation function by the spectral representation by cosine series for small sample sizes. More accurate results can be achieved for Modal decomposition and Cholesky decomposition by increasing the number of sample sizes but in detriment of the required CPU time. As indicated in Table 1, for small sample sizes, the three methods are approximately equivalent in terms of CPU time requirements. However, for large samples size, in order to obtain more accurate results, the advantage of Spectral representation is evident with respect to the other methods.

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## REFERENCES

Ripley, Brain D., 1987, Stochastic Simulation. Willey series in probability and mathematical statistics. 237p.

Liu, P.-L.; Der Kiureghian, A., 1986, Multivariate distribution models with prescribed marginal and covariances. Probabilistic Engineering Mechanics, Vol.1, No.2.

Liu, P.-L.; Liu, K.-G., 1993, Selection of random field mesh in finite element reliability analysis. Journal of Engineering Mechanics, Vol. 199, No. 4, April.

Shinozuka, M.; Deodatis, G., 1996, Simulation of multi-dimensional Gaussian stochastic fields by spectral representation. Applied Mechanics, Vol. 49, No. 1, January.

Yamazaki, F.; Shinozuka, M., 1988, Digital generation of non-Gaussian stochastic fields. Journal of Engineering Mechanics, Vol. 114, No. 7, July.

Yamazaki, F.; Shinozuka, M.,1990, Simulation of stochastic fields by statistical preconditioning. Journal of Engineering Mechanics, Vol. 116, No.2, February.